

# ANALYTICAL SOLUTION OF THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR CONVEX QUADRILATERALS IN THE EUCLIDEAN PLANE: THE CASE OF TWO PAIRS OF EQUAL WEIGHTS

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ABSTRACT. The weighted Fermat-Torricelli problem for four non-collinear points in  $\mathbb{R}^2$  states that:

Given four non-collinear points  $A_1, A_2, A_3, A_4$  and a positive real number (weight)  $B_i$  which correspond to each point  $A_i$ , for  $i = 1, 2, 3, 4$ , find a fifth point such that the sum of the weighted distances to these four points is minimized. We present an analytical solution for the weighted Fermat-Torricelli problem for convex quadrilaterals in  $\mathbb{R}^2$  for the following two cases:

(a)  $B_1 = B_2$  and  $B_3 = B_4$ , for  $B_1 > B_4$  and (b)  $B_1 = B_3$  and  $B_2 = B_4$ .

## 1. INTRODUCTION

The weighted Fermat-Torricelli problem for  $n$  non-collinear points in  $\mathbb{R}^2$  refers to finding the unique point  $A_0 \in \mathbb{R}^2$ , minimizing the objective function:

$$f(X) = \sum_{i=1}^n B_i \|X - A_i\|,$$

$X \in \mathbb{R}^2$  given four non-collinear points  $\{A_1, A_2, A_3, A_4, \dots, A_n\}$  with corresponding positive real numbers (weights)  $B_1, B_2, B_3, B_4, \dots, B_n$  where  $\|\cdot\|$  denotes the Euclidean distance.

The existence and uniqueness of the weighted Fermat-Torricelli point and a complete characterization of the solution of the weighted Fermat-Torricelli problem has been given by Y. S Kupitz and H. Martini (see [5], theorem 1.1, reformulation 1.2 page 58, theorem 8.5 page 76, 77). A particular case of this result for four non-collinear points in  $\mathbb{R}^2$ , is given by the following theorem:

**Theorem 1.** [2],[5] *Let there be given four non-collinear points  $\{A_1, A_2, A_3, A_4\}$ ,  $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$  with corresponding positive weights  $B_1, B_2, B_3, B_4$ .*

(a) *The weighted Fermat-Torricelli point  $A_0$  exists and is unique.*

(b) *If for each point  $A_i \in \{A_1, A_2, A_3, A_4\}$*

$$\left\| \sum_{j=1, j \neq i}^4 B_j \vec{u}(A_i, A_j) \right\| > B_i, \quad (1.1)$$

*for  $i, j = 1, 2, 3$  holds, then*

(b<sub>1</sub>) *the weighted Fermat-Torricelli point  $A_0$  (weighted floating equilibrium point)*

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does not belong to  $\{A_1, A_2, A_3, A_4\}$  and  
( $b_2$ )

$$\sum_{i=1}^4 B_i \vec{u}(A_0, A_i) = \vec{0}, \quad (1.2)$$

where  $\vec{u}(A_k, A_l)$  is the unit vector from  $A_k$  to  $A_l$ , for  $k, l \in \{0, 1, 2, 3, 4\}$  (Weighted Floating Case).

(c) If there is a point  $A_i \in \{A_1, A_2, A_3, A_4\}$  satisfying

$$\left\| \sum_{j=1, j \neq i}^4 B_j \vec{u}(A_i, A_j) \right\| \leq B_i, \quad (1.3)$$

then the weighted Fermat-Torricelli point  $A_0$  (weighted absorbed point) coincides with the point  $A_i$  (Weighted Absorbed Case).

In 1969, E. Cockayne, Z. Melzak proved in [3] by using Galois theory that for a specific set of five non-collinear points the unweighted Fermat-Torricelli point  $A_0$  cannot be constructed by ruler and compass in a finite number of steps (Euclidean construction).

In 1988, C. Bajaj also proved in [1] by applying Galois theory that for  $n \geq 5$  the weighted Fermat-Torricelli problem for  $n$  non-collinear points is in general not solvable by radicals over the field of rationals in  $\mathbb{R}^3$ .

We recall that for  $n = 4$ , Fagnano proved that the solution of the unweighted Fermat-Torricelli problem ( $B_1 = B_2 = B_3 = B_4$ ) for convex quadrilaterals in  $\mathbb{R}^2$  is the intersection point of the two diagonals and it is well known that the solution of the weighted Fermat-Torricelli problem for non-convex quadrilaterals is the vertex of the non-convex angle. Extensions of Fagnano result to some metric spaces are given by Plastria in [6].

In 2012, Roussos studied the unweighted Fermat-Torricelli problem for Euclidean triangles and Uteshev studied the corresponding weighted Fermat-Torricelli problem and succeeded in finding an analytic solution by using some algebraic system of equations (see [7] and [9]).

Thus, we consider the following open problem:

**Problem 1.** Find an analytic solution with respect to the weighted Fermat-Torricelli problem for convex quadrilaterals in  $\mathbb{R}^2$ , such that the corresponding weighted Fermat-Torricelli point is not any of the given points.

In this paper, we present an analytic solution for the weighted Fermat-Torricelli problem for a given tetragon in  $\mathbb{R}^2$  for  $B_1 > B_4$ ,  $B_1 = B_2$  and  $B_3 = B_4$ , by expressing the objective function as a function of the linear segment which connects the intersection point of the two diagonals and the corresponding weighted Fermat-Torricelli point (Section 2, Theorem 2).

By expressing the angles  $\angle A_1 A_0 A_2$ ,  $\angle A_2 A_0 A_3$ ,  $\angle A_3 A_0 A_4$  and  $\angle A_4 A_0 A_1$  as a function of  $B_1$ ,  $B_4$  and  $a$  and taking into account the invariance property of the weighted Fermat-Torricelli point, we obtain an analytic solution for a convex quadrilateral having the same weights with the tetragon (Section 3, Theorem 3).

Finally, we derive that the solution for the weighted Fermat-Torricelli problem for a given convex quadrilateral in  $\mathbb{R}^2$  for the weighted floating case for  $B_1 = B_3$

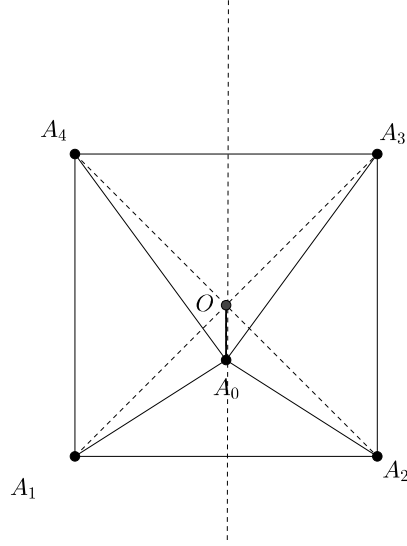


FIGURE 1. The weighted Fermat-Torricelli problem for a tetragon  $B_1 = B_2$  and  $B_3 = B_4$  for  $B_1 > B_4$

and  $B_2 = B_4$  is the intersection point (Weighted Fermat-Torricelli point) of the two diagonals (Section 4, Theorem 4).

2. THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR A TETRAGON: THE CASE  $B_1 = B_2$  AND  $B_3 = B_4$ .

We consider the weighted Fermat-Torricelli problem for a tetragon  $A_1A_2A_3A_4$ , for  $B_1 > B_4$ ,  $B_1 = B_2$  and  $B_3 = B_4$ .

We denote by  $a_{ij}$  the length of the linear segment  $A_iA_j$ ,  $O$  the intersection point of  $A_1A_3$  and  $A_2A_4$ ,  $y$  the length of the linear segment  $OA_0$  and  $\alpha_{ikj}$  the angle  $\angle A_iA_kA_j$  for  $i, j, k = 0, 1, 2, 3, 4, i \neq j \neq k$  (See fig. 1) and we set  $a_{12} = a_{23} = a_{34} = a_{41} = a$ .

**Problem 2.** *Given a tetragon  $A_1A_2A_3A_4$  and a weight  $B_i$  which corresponds to the vertex  $A_i$ , for  $i = 1, 2, 3, 4$ , find a fifth point  $A_0$  (weighted Fermat-Torricelli point) which minimizes the objective function*

$$f = B_1a_{01} + B_2a_{02} + B_3a_{03} + B_4a_{04} \quad (2.1)$$

for  $B_1 > B_4$ ,  $B_1 = B_2$  and  $B_3 = B_4$ .

**Theorem 2.** *The location of the weighted Fermat-Torricelli point of  $A_1A_2A_3A_4$  for  $B_1 = B_2$ ,  $B_3 = B_4$  and  $B_1 > B_4$  is given by:*

$$y = \frac{1}{2} \sqrt{\frac{a^2}{4} + r} - \frac{1}{2} \sqrt{\frac{a^2}{4} - \frac{t^{1/3}}{24 \cdot 2^{1/3} q^{1/3}} - \frac{25pq^{1/3}}{3 \cdot 2^{2/3} t^{1/3} (B_1^2 - B_4^2)^2} + \frac{a^2 B_1^2 - a^2 B_4^2}{12 (B_1^2 - B_4^2)} - \frac{-a^3 B_1^2 - a^3 B_4^2}{2 \sqrt{\frac{a^2}{4} + r} (B_1^2 - B_4^2)}} \quad (2.2)$$

where

$$t = 2000a^6 B_1^6 - 2544a^6 B_1^4 B_4^2 + 2544a^6 B_1^2 B_4^4 - 2000a^6 B_4^6 + 192\sqrt{3} \sqrt{a^{12} B_1^2 B_4^2 (B_1^2 - B_4^2)^2 (125B_1^4 - 142B_1^2 B_4^2 + 125B_4^4)}, \quad (2.3)$$

$$p = a^4 B_1^4 - 2a^4 B_1^2 B_4^2 + a^4 B_4^4, \quad (2.4)$$

$$q = B_1^6 - 3B_1^4 B_4^2 + 3B_1^2 B_4^4 - B_4^6 \quad (2.5)$$

and

$$r = \frac{t^{1/3}}{24 \cdot 2^{1/3} q^{1/3}} + \frac{25pq^{1/3}}{3 \cdot 2^{2/3} t^{1/3} (B_1^2 - B_4^2)^2} - \frac{a^2 B_1^2 - a^2 B_4^2}{12 (B_1^2 - B_4^2)}. \quad (2.6)$$

*Proof of Theorem 2:* Taking into account the symmetry of the weights  $B_1 = B_4$  and  $B_2 = B_3$  for  $B_1 > B_4$  and the symmetries of the tetragon the objective function (2.15) of the weighted Fermat-Torricelli problem (Problem 2) could be reduced to an equivalent Problem by placing a wall to the midperpendicular line from  $A_1A_2$  and  $A_3A_4$  which states that: Find a point  $A_0$  which belongs to the midperpendicular of  $A_1A_2$  and  $A_3A_4$  and minimizes the objective function

$$\frac{f}{2} = B_1 a_{01} + B_4 a_{04}. \quad (2.7)$$

We express  $a_{01}$ ,  $a_{02}$ ,  $a_{03}$  and  $a_{04}$  as a function of  $y$  :

$$a_{01}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2 \quad (2.8)$$

$$a_{02}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2 \quad (2.9)$$

$$a_{03}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} + y\right)^2 \quad (2.10)$$

$$a_{04}^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} + y\right)^2 \quad (2.11)$$

By replacing (2.8) and (2.11) in (2.7) we get:

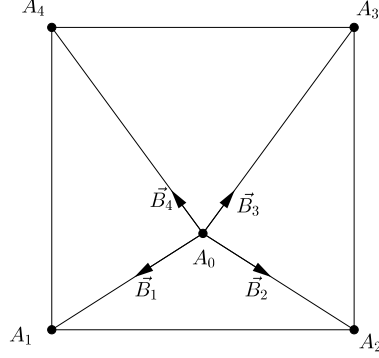


FIGURE 2. The weighted floating equilibrium point (weighted Fermat-Torricelli point)  $A_0$  for a tetragon  $B_1 = B_2$  and  $B_3 = B_4$  for  $B_1 > B_4$

$$B_1 \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2} + B_4 \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} + y\right)^2} \rightarrow \min. \quad (2.12)$$

By differentiating (2.12) with respect to  $y$ , and by squaring both parts of the derived equation, we get:

$$\frac{B_1^2 \left(\frac{a}{2} - y\right)^2}{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2} = \frac{B_4^2 \left(\frac{a}{2} + y\right)^2}{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} + y\right)^2} \quad (2.13)$$

or

$$8(B_1^2 - B_4^2)y^4 + 2a^2(-B_1^2 + B_4^2)y^2 - 2a^3(B_1^2 + B_4^2)y + a^4(B_1^2 - B_4^2) = 0. \quad (2.14)$$

By solving the fourth order equation with respect to  $y$ , we derive two complex solutions and two real solutions (Ferrari's solution, see also in [8]) which depend on  $B_1, B_4$  and  $a$ . One of the two real solutions with respect to  $y$  is (2.2). From (2.2), we obtain that the weighted Fermat-Torricelli point  $A_0$  is located at the interior of  $A_1A_2A_3A_4$  (see fig. 2).

□

The Complementary Fermat-Torricelli problem was stated by Courant and Robbins (see in [4, pp. 358]) for a triangle which is derived by the weighted Fermat-Torricelli problem by placing one negative weight to one of the vertices of the triangle and asks for the complementary weighted Fermat-Torricelli point which minimizes the corresponding objective function.

We need to state the Complementary weighted Fermat-Torricelli problem for a tetragon, in order to explain the second real solution which have been obtained by (2.14) with respect to  $y$ .

**Problem 3.** *Given a tetragon  $A_1A_2A_3A_4$  and a weight  $B_i$  (a positive or negative real number) which corresponds to the vertex  $A_i$ , for  $i = 1, 2, 3, 4$ , find a fifth point  $A_0$  (weighted Fermat-Torricelli point) which minimizes the objective function*

$$f = B_1a_{01} + B_2a_{02} + B_3a_{03} + B_4a_{04} \quad (2.15)$$

for  $\|B_1\| > \|B_4\|$ ,  $B_1 = B_2$  and  $B_3 = B_4$ .

**Proposition 1.** *The location of the complementary weighted Fermat-Torricelli point  $A'_0$  (solution of Problem 3) of  $A_1A_2A_3A_4$  for  $B_1 = B_2 < 0$ ,  $B_3 = B_4 < 0$  and  $\|B_1\| > \|B_4\|$  coincides with the location of the corresponding weighted Fermat-Torricelli point of  $A_1A_2A_3A_4$  for  $B_1 = B_2 > 0$ ,  $B_3 = B_4 > 0$  and  $\|B_1\| > \|B_4\|$ .*

*Proof of Proposition 1:* By applying theorem 2 for  $B_1 = B_2 < 0$ ,  $B_3 = B_4 < 0$  we derive the weighted floating equilibrium condition (see fig. 3):

$$\vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = \vec{0} \quad (2.16)$$

or

$$(-\vec{B}_1) + (-\vec{B}_2) + (-\vec{B}_3) + (-\vec{B}_4) = \vec{0}. \quad (2.17)$$

From (2.16) and (2.17), we derive that the complementary weighted Fermat-Torricelli point  $A'_0$  coincides with the weighted Fermat-Torricelli point  $A_0$ . The difference between the figures 2 and 3 is that the vectors  $\vec{B}_i$  change direction from  $A_i$  to  $A_0$ , for  $i = 1, 2, 3, 4$ . □

**Proposition 2.** *The location of the complementary weighted Fermat-Torricelli point  $A'_0$  (solution of Problem 3) of  $A_1A_2A_3A_4$  for  $B_1 = B_2 < 0$ ,  $B_3 = B_4 > 0$  or  $B_1 = B_2 > 0$ ,  $B_3 = B_4 < 0$  and  $\|B_1\| > \|B_4\|$  is given by:*

$$y = \frac{\sqrt{d}}{2} + \frac{1}{2} \sqrt{-\frac{\frac{2}{(\sqrt{s}+z)^{1/3}} + 2^{2/3}(\sqrt{s}+z)^{1/3} + 32a\left(2 + a\left(-2 - \frac{3}{\sqrt{d}}\right)\right)B_1^2 + 32a\left(-2 + 2a - \frac{3a}{\sqrt{d}}\right)B_4^2}{96(B_1^2 - B_4^2)}}. \quad (2.18)$$

where

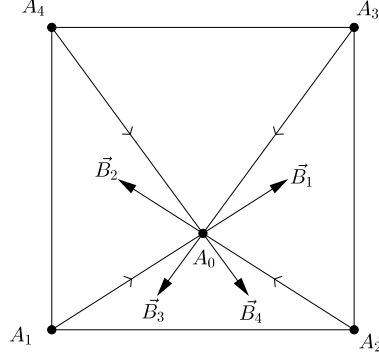


FIGURE 3. The complementary weighted Fermat-Torricelli point  $A'_0$  for a tetragon  $B_1 = B_2 < 0$  and  $B_3 = B_4 < 0$  for  $\|B_1\| > \|B_4\|$

$$z = -1024 (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2)^3 + 27648 (B_1^2 - B_4^2) (a^2B_1^2 + a^2B_4^2)^2 + 9216 (B_1^2 - B_4^2) (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2) (2a^3B_1^2 + a^4B_1^2 - 2a^3B_4^2 - a^4B_4^2) \quad (2.19)$$

$$w = 64 (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2)^2 + 192 (B_1^2 - B_4^2) (2a^3B_1^2 + a^4B_1^2 - 2a^3B_4^2 - a^4B_4^2), \quad (2.20)$$

$$s = -4w^3 + (-1024 (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2)^3 + 27648 (B_1^2 - B_4^2) (a^2B_1^2 + a^2B_4^2)^2 + 9216 (B_1^2 - B_4^2) (-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2) (2a^3B_1^2 + a^4B_1^2 - 2a^3B_4^2 - a^4B_4^2))^2 \quad (2.21)$$

and

$$d = \frac{1}{2} (-a + a^2) + \frac{w}{24 \cdot 2^{2/3} (\sqrt{s} + z)^{1/3} (B_1^2 - B_4^2)} + \frac{(\sqrt{s} + z)^{1/3}}{48 \cdot 2^{1/3} (B_1^2 - B_4^2)} - \frac{-aB_1^2 + a^2B_1^2 + aB_4^2 - a^2B_4^2}{6 (B_1^2 - B_4^2)}. \quad (2.22)$$

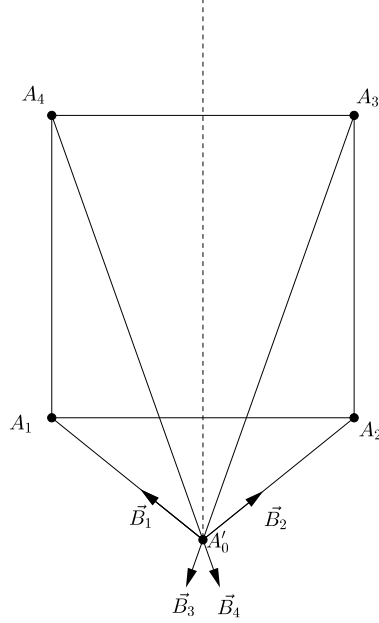


FIGURE 4. The complementary weighted Fermat-Torricelli point  $A'_0$  for a tetragon  $B_1 = B_2 > 0$  and  $B_3 = B_4 < 0$  for  $\|B_1\| > \|B_4\|$

*Proof of Proposition 2:* Taking into account (2.12) for  $B_1 = B_2 < 0$ ,  $B_3 = B_4 > 0$  or  $B_1 = B_2 > 0$ ,  $B_3 = B_4 < 0$  and  $\|B_1\| > \|B_4\|$  and differentiating (2.12) with respect to  $y \equiv OA'_0$ , and by squaring both parts of the derived equation, we obtain (2.14) which is a fourth order equation with respect to  $y$ . The second real solution of  $y$  gives (2.18). From (2.18) and the vector equilibrium condition  $\vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = \vec{0}$  we obtain that the complementary weighted Fermat-Torricelli point  $A'_0$  for  $B_1 = B_2 < 0$ ,  $B_3 = B_4 > 0$  coincides with the complementary weighted Fermat-Torricelli point  $A''_0$  for  $B_1 = B_2 > 0$ ,  $B_3 = B_4 < 0$  ( Fig. 4 and 5). Furthermore, the solution (2.18) yields that the complementary  $A'_0$  is located outside the tetragon  $A_1A_2A_3A_4$  (Fig. 4 and 5).  $\square$

**Example 1.** Given a tetragon  $A_1A_2A_3A_4$  in  $\mathbb{R}^2$ ,  $a = 2$ ,  $B_1 = B_2 = 1.5$ ,  $B_3 = B_4 = 1$  from 2.2 and (2.18) we get  $y = 0.36265$  and  $y = 1.80699$ , respectively, with five digit precision. The weighted Fermat-Torricelli point  $A_0$  and the complementary weighted Fermat-Torricelli point  $A'_0 \equiv A_0$  for  $B_1 = B_2 = -1.5$  and  $B_3 = B_4 = -1$  corresponds to  $y = 0.36265$ . The complementary weighted Fermat-Torricelli point  $A'_0$  for  $B_1 = B_2 = -1.5$  and  $B_3 = B_4 = 1$  or  $B_1 = B_2 = 1.5$  and  $B_3 = B_4 = -1$  lies outside the tetragon  $A_1A_2A_3A_4$  and corresponds to  $y = 1.80699$



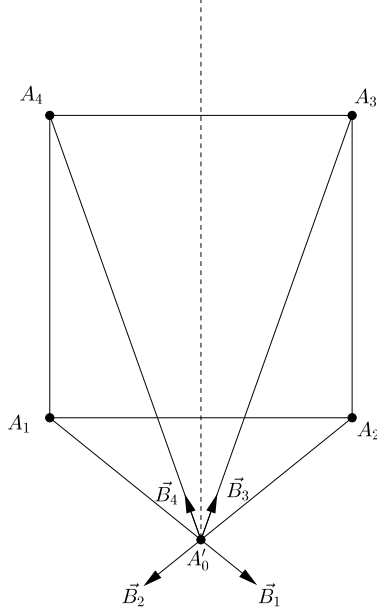


FIGURE 5. The complementary weighted Fermat-Torricelli point for a tetragon  $B_1 = B_2 < 0$  and  $B_3 = B_4 > 0$  for  $\|B_1\| > \|B_4\|$

We denote by  $A_{12}$  the intersection point of the midperpendicular of  $A_1A_2$  and  $A_3A_4$  with  $A_1A_2$  and by  $A_{14}$  the intersection point of the perpendicular from  $A_0$  to the line defined by  $A_1A_4$ .

We shall calculate the angles  $\alpha_{102}, \alpha_{203}, \alpha_{304}$  and  $\alpha_{401}$ .

**Proposition 3.** *The angles  $\alpha_{102}, \alpha_{203}, \alpha_{304}$  and  $\alpha_{401}$  are given by:*

$$\alpha_{102} = 2 \arccos \frac{\frac{a}{2} - y(B_1, B_4, a)}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} - y\right)^2}}, \quad (2.23)$$

$$\alpha_{304} = 2 \arccos \left( \frac{B_1}{B_4} \cos \frac{\alpha_{102}}{2} \right) \quad (2.24)$$

and

$$\alpha_{401} = \alpha_{203} = \pi - \frac{\alpha_{102}}{2} - \frac{\alpha_{304}}{2}. \quad (2.25)$$

*Proof of Proposition 3:* From  $\triangle A_1A_{12}A_0$  and taking into account (2.2), we get (2.23).

From the right angled triangles  $\triangle A_1A_{12}A_0$ ,  $\triangle A_1A_{14}A_0$  and  $\triangle A_4A_{14}A_0$ , we obtain:

$$a_{01} = \frac{a}{2 \sin \frac{\alpha_{102}}{2}}, \quad (2.26)$$

$$a_{04} = \frac{a}{2 \sin \frac{\alpha_{304}}{2}}, \quad (2.27)$$

and

$$a_{01} \cos \frac{\alpha_{102}}{2} + a_{04} \cos \frac{\alpha_{304}}{2} = a, \quad (2.28)$$

By dividing both members of (2.28) by (2.26) or (2.27), we get:

$$\cot \frac{\alpha_{102}}{2} = 2 - \cot \frac{\alpha_{304}}{2}. \quad (2.29)$$

From (2.29) the angle  $\alpha_{102}$  is expressed as a function of  $\alpha_{304}$  :  $\alpha_{102} = \alpha_{102}(\alpha_{304})$ .  
By replacing (2.26) and (2.27) in (2.7) we get:

$$\frac{B_1}{\sin \frac{\alpha_{102}}{2}} + \frac{B_4}{\sin \frac{\alpha_{304}}{2}} \rightarrow \min. \quad (2.30)$$

By differentiating (2.29) with respect to  $\alpha_{304}$ , we derive:

$$\frac{d\alpha_{102}}{d\alpha_{304}} = -\frac{\sin^2 \frac{\alpha_{102}}{2}}{\sin^2 \frac{\alpha_{304}}{2}}. \quad (2.31)$$

By differentiating (2.30) with respect to  $\alpha_{304}$  and replacing in the derived equation (2.31) we obtain (2.24).

From the equality of triangles  $\triangle A_1 A_0 A_4$  and  $\triangle A_2 A_0 A_3$ , we get  $\alpha_{401} = \alpha_{203}$  which yields (2.25). □

### 3. THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR CONVEX QUADRILATERALS: THE CASE $B_1 = B_2$ AND $B_3 = B_4$ .

We need the following lemma, in order to find the weighted Fermat-Torricelli point for a given convex quadrilateral  $A'_1 A'_2 A'_3 A'_4$  in  $\mathbb{R}^2$ , which has been proved in [10, Proposition 3.1, pp. 414] for convex polygons in  $\mathbb{R}^2$ .

**Lemma 1.** [10, Proposition 3.1, pp. 414] *Let  $A_1 A_2 A_3 A_4$  be a tetragon in  $\mathbb{R}^2$  and each vertex  $A_i$  has a non-negative weight  $B_i$  for  $i = 1, 2, 3, 4$ . Assume that the floating case of the weighted Fermat-Torricelli point  $A_0$  is valid:*

$$\left\| \sum_{j=1, i \neq j}^4 B_j \vec{u}(A_i, A_j) \right\| > B_i. \quad (3.1)$$

*If  $A_0$  is connected with every vertex  $A_i$  for  $i = 1, 2, 3, 4$  and a point  $A'_i$  is selected with corresponding non-negative weight  $B_i$  on the ray that is defined by the line segment  $A_0 A_i$  and the convex quadrilateral  $A'_1 A'_2 A'_3 A'_4$  is constructed such that:*

$$\left\| \sum_{j=1, i \neq j}^4 B_j \vec{u}(A'_i, A'_j) \right\| > B_i, \quad (3.2)$$

*then the weighted Fermat-Torricelli point  $A'_0$  of  $A'_1 A'_2 A'_3 A'_4$  is identical with  $A_0$ .*

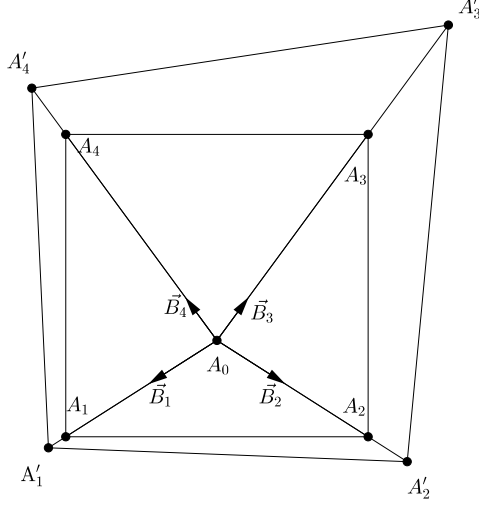


FIGURE 6. The weighted Fermat-Torricelli point of a convex quadrilateral for  $B_1 = B_2$ ,  $B_3 = B_4$  and  $B_1 > B_4$

Let  $A'_1 A'_2 A'_3 A'_4$  be a convex quadrilateral with corresponding non-negative weights  $B_1 = B_2$  at the vertices  $A'_1, A'_2$  and  $B_3 = B_4$  at the vertices  $A'_3, A'_4$ .

We select  $B_1$  and  $B_4$  which satisfy the inequalities (3.1), (3.2) and  $B_1 > B_4$ , which correspond to the weighted floating case of the tetragon  $A_1 A_2 A_3 A_4$  and  $A'_1 A'_2 A'_3 A'_4$ . Furthermore, we assume that  $A_0$  is located at the interior of  $\triangle A'_1 A'_2 A'_3$ .

We denote by  $a'_{ij}$  the length of the linear segment  $A'_i A'_j$ ,  $\alpha'_{ikj}$  the angle  $\angle A'_i A'_k A'_j$  for  $i, j, k = 0, 1, 2, 3, 4, i \neq j \neq k$  (See fig. 6)

**Theorem 3.** *The location of the weighted Fermat-Torricelli point  $A_0$  of  $A'_1 A'_2 A'_3 A'_4$  for  $B_1 = B_2$  and  $B_3 = B_4$  under the conditions (3.1), (3.2) and  $B_1 > B_4$ , is given by:*

$$a'_{02} = a'_{12} \frac{\sin(\alpha'_{213} - \alpha'_{013})}{\sin \alpha_{102}} \quad (3.3)$$

and

$$\alpha'_{120} = \pi - \alpha_{102} - (\alpha'_{123} - \alpha'_{013}), \quad (3.4)$$

where

$$\alpha'_{013} = \frac{\sin(\alpha'_{213}) - \cos(\alpha'_{213}) \cot(\alpha_{102}) - \frac{a'_{31}}{a'_{12}} \cot(\alpha_{304} + \alpha_{401})}{-\cos(\alpha'_{213}) - \sin(\alpha'_{213}) \cot(\alpha_{102}) + \frac{a'_{31}}{a'_{12}}} \quad (3.5)$$

and

$$\cot(\alpha_{304} + \alpha_{401}) = \frac{B_1 + B_2 \cos(\alpha_{102}) + B_4 \cos(\alpha_{401})}{B_4 \sin(\alpha_{401}) - B_2 \sin(\alpha_{102})}. \quad (3.6)$$

*Proof of Theorem 3:* From lemma 1 the weighted Fermat-Torricelli point  $A_0$  of  $A_1 A_2 A_3 A_4$  is the same with the weighted Fermat-Torricelli point  $A'_0 \equiv A_0$  of  $A'_1 A'_2 A'_3 A'_4$ , for the weights  $B_1 = B_2$  and  $B_3 = B_4$ , under the conditions (3.1), (3.2) and  $B_1 > B_4$ .

Thus, we derive that:

$$\alpha_{102} = \alpha'_{102}, \alpha_{203} = \alpha'_{203}, \alpha_{304} = \alpha'_{304} \text{ and } \alpha_{401} = \alpha'_{401}.$$

By applying the same technique that was used in [10, Solution 2.2, pp. 412-414] we express  $a'_{02}$ ,  $a'_{03}$ ,  $a'_{04}$  as a function of  $a'_{01}$  and  $\alpha'_{013}$  taking into account the cosine law to the corresponding triangles  $\triangle A'_2 A'_1 A'_0$ ,  $\triangle A'_3 A'_1 A'_0$  and  $\triangle A'_4 A'_1 A'_0$ . By differentiating the objective function (2.15) with respect to  $a'_{01}$  and  $\alpha'_{013}$  and applying the sine law in  $\triangle A'_2 A'_1 A'_0$ ,  $\triangle A'_3 A'_1 A'_0$  and  $\triangle A'_4 A'_1 A'_0$  we derive (3.6) and solving with respect to  $\alpha'_{013}$  we derive (3.5). By applying the sine law in  $\triangle A'_2 A'_1 A'_0$ , we get (3.3).

Finally,  $\alpha'_{120} = \pi - \alpha_{102} - (\alpha'_{123} - \alpha'_{013})$ .

□

#### 4. THE WEIGHTED FERMAT-TORRICELLI PROBLEM FOR CONVEX QUADRILATERALS: THE CASE $B_1 = B_3$ AND $B_2 = B_4$ .

Let  $A'_1 A'_2 A'_3 A'_4$  be a convex quadrilateral with corresponding non-negative weights  $B_1 = B_3$  at the vertices  $A'_1, A'_2$  and  $B_2 = B_4$  at the vertices  $A'_3, A'_4$ .

We select  $B_1$  and  $B_4$  which satisfy the inequalities (3.1), such that  $A_0$  is an interior point of  $A'_1 A'_2 A'_3 A'_4$ .

**Theorem 4.** *The location of the weighted Fermat-Torricelli point  $A_0$  of  $A'_1 A'_2 A'_3 A'_4$  for  $B_1 = B_3$  and  $B_2 = B_4$  under the conditions (3.1), (3.2) is the intersection point of the diagonals  $A'_1 A'_3$  and  $A'_2 A'_4$ .*

*Proof of Theorem 4:* From the weighted floating equilibrium condition (1.2) of theorem 1 we get:

$$\vec{B}_1 + \vec{B}_2 = -(\vec{B}_3 + \vec{B}_4) \quad (4.1)$$

and

$$\vec{B}_1 + \vec{B}_4 = -(\vec{B}_2 + \vec{B}_3) \quad (4.2)$$

Taking the inner product of the first part of (4.1) with  $\vec{B}_1 + \vec{B}_2$  and the second part of (4.1) with  $-(\vec{B}_3 + \vec{B}_4)$ , we derive that:

$$\alpha_{102} = \alpha_{304}.$$

Similarly, taking the inner product of the first part of (4.2) with  $\vec{B}_1 + \vec{B}_4$  and the second part of (4.2) with  $-(\vec{B}_2 + \vec{B}_3)$ , we derive that:

$$\alpha_{104} = \alpha_{203}.$$

□

**Proposition 4.** *The location of the complementary weighted Fermat-Torricelli point  $A_0$  of  $A'_1A'_2A'_3A'_4$  for  $B_1 = B_3 < 0$  and  $B_2 = B_4 < 0$  under the conditions (3.1), (3.2) is the intersection point of the diagonals  $A'_1A'_3$  and  $A'_2A'_4$ .*

*Proof of Proposition 4:* Taking into account (2.15) for  $B_1 = B_3 < 0$ ,  $B_2 = B_4 < 0$  we derive the same vector equilibrium condition  $\vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = \vec{0}$ . Therefore, we obtain that the complementary weighted Fermat-Torricelli point  $A'_0$  for  $B_1 = B_3 < 0$ ,  $B_2 = B_4 < 0$  coincides with the weighted Fermat-Torricelli point  $A_0$  of  $A'_1A'_2A'_3A'_4$  for  $B_1 = B_3 > 0$ ,  $B_2 = B_4 > 0$ . □

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